Symmetry, Scaling, and Optimal Time–Frequency Concentration: Minimising the Heisenberg Uncertainty in Piecewise-Polynomial and Wavelet Dictionaries

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We study optimal joint localisation in time and frequency through a variational formulation of the Heisenberg uncertainty product. Starting from a hierarchy of piecewise-polynomial function classes on a fixed compact interval, we define suitable mean, variance and uncertainty operators and show that the resulting functional is invariant under translations and changes of scale. This makes it possible to compare time-frequency atoms independently of their location and dilation. We then prove a symmetry reduction result: the infimum of the uncertainty over the non-negative endpoint-vanishing class is already reached inside its even subclass. Within two concrete wavelet dictionaries generated by compactly supported polynomial profiles, the tent function is shown to be the unique minimiser, with uncertainty value

$$U = \frac{3}{10}.$$

Finally, we analyse the family of p-fold self-convolutions of the rectangular window, $\mathrm{rect}^{\{p\}}$, and prove that their uncertainty decreases monotonically to the sharp Heisenberg bound $1/4$ as $p \to \infty$. The results provide a constructive and invariance-aware explanation of why tent functions and high-order spline-like atoms arise naturally as well-localised windows, and suggest design principles for wavelet and Gabor-type dictionaries with provably controlled time-frequency concentration.

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