Mathematically rigorous justification of the Reynolds equation for a spherical bearing

Guy Bayada, José M. Rodríguez, Raquel Taboada-Vázquez

We prove that the solution of the Stokes problem in a domain between two spheres converges, as the separation between them tends to zero, to the solution (unique up to the addition of a constant) of the following Reynolds problem:

$$\frac{h^2}{12\mu}\frac{\partial p}{\partial \varphi} = -Rk_0 \text{ on } \varphi = \varphi_0, \tag{1}$$

$$\frac{h^2}{12\mu}\frac{\partial p}{\partial \varphi} = -Rk_1 \text{ on } \varphi = \varphi_1, \tag{2}$$

$$\frac{\partial}{\partial \varphi}\left(\frac{h^3 \sin\varphi}{12\mu}\frac{\partial p}{\partial \varphi}\right) + \frac{\partial}{\partial \theta}\left(\frac{h^3}{12\mu \sin\varphi}\frac{\partial p}{\partial \theta}\right) = \frac{R^2\omega \sin\varphi}{2}\frac{\partial h}{\partial \theta} \tag{3}$$

where

$$\vec{X}(\varphi, \theta) = R(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), \tag{4}$$

$$\varphi \in (\varphi_0, \varphi_1),\ 0 < \varphi_0 < \varphi_1 < \pi,\ \theta \in (0, 2\pi), \tag{5}$$

is the parameterization of the inner sphere, R is its radius, $p(\varphi, \theta)$ is the pressure, $h(\varphi, \theta)$ gives the distance between the two spheres, $k_0(\theta)$ and $k_1(\theta)$ are the averaged input/output flows at $\varphi = \varphi_0$ and $\varphi = \varphi_1$, respectively, ω is the angular velocity of the inner sphere (we assume the outer sphere is stationary) and μ is the dynamic viscosity.

To our knowledge, this is the first mathematically rigorous justification of the Reynolds equation for a spherical bearing.

Acknowledgements

This work has been partially supported by Project PID2024-158035NB-I00, funded by MICIU/AEI/10.13039/501100011033 and FEDER, EU.

References

[1] G. Bayada, M. Chambat, *The transition between the Stokes equation and the Reynolds equation: a mathematical proof*, Appl. Math. Opt. 14, 73–93 (1986). `https://doi.org/10.1007/BF01442229`.

[2] G. Bayada, J. M. Rodríguez, R. Taboada-Vázquez, *Asymptotic justification of the Reynolds equation for a spherical bearing*, to appear. `https://doi.org/10.48550/arXiv.2606.30148`.

[3] O. Reynolds, *On the theory of lubrication and its application to Mr. Beauchamp tower's experiment, including the determination of the viscosity of olive oil*, Phil. Trans. Roy. Soc., Paper 52 (1886). `https://doi.org/10.1098/rstl.1886.0005`.

First Author: Guy Bayada
Affiliation: *INSA Lyon, Université Lyon 1, CNRS, ICJ UMR5208*
69621 Villeurbanne, France
e-mail: `guy.bayada@insa-lyon.fr`

Second Author: José M. Rodríguez
Affiliation 1: *Department of Mathematics, Universidade da Coruña*
15071 A Coruña, Spain
Affiliation 2: *CITMAga, 15782 Santiago de Compostela, Spain*
e-mail: `jose.rodriguez.seijo@udc.es`

Third Author: Raquel Taboada-Vázquez
Affiliation 1: *Department of Mathematics, Universidade da Coruña*
15071 A Coruña, Spain
Affiliation 2: *CITMAga, 15782 Santiago de Compostela, Spain*
e-mail: `raquel.taboada@udc.es`