Sufficient Conditions for Finite-Time and Fixed-time Stability of Time-Varying Differential Equations Using Dual Dynamic Programming

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In this work, we investigate the finite-time and fixed-time stability of impulsive differential equations by reformulating the stability analysis problem as an optimal control problem. This transformation provides a unified framework that enables the use of powerful tools from optimal control theory for the study of stability properties in impulsive dynamical systems. Building upon this formulation, we develop dual dynamic methodologies together with corresponding verification theorems, which allow us to derive rigorous and easily verifiable sufficient conditions ensuring both finite-time and fixed-time stability.

A central contribution of the proposed approach is the explicit treatment of impulsive effects as control variables within the optimization framework. Unlike conventional methods, where impulses are typically regarded as predefined system components or external perturbations, our formulation incorporates them directly into the control structure. This perspective not only broadens the analytical framework but also provides greater flexibility in characterizing and influencing system behavior. At the same time, the inclusion of impulses as decision variables introduces additional mathematical challenges, since the interaction between continuous dynamics and discontinuous state jumps must be carefully addressed in both the theoretical development and the stability analysis.

To overcome these difficulties, we establish a set of dual dynamic principles and verification theorems that connect the optimal control formulation with the desired stability properties. These results offer new insights into the mechanisms through which impulsive actions affect convergence rates and stability guarantees. Furthermore, the developed framework highlights the distinct roles that continuous control actions and impulsive interventions play in achieving finite-time and fixed-time stabilization.

The effectiveness and practical relevance of the proposed methodology are illustrated through representative examples. These examples show that impulses are not merely auxiliary features of the system but can be essential components in preserving and enhancing stability. In particular, they demonstrate how appropriately designed impulsive actions can significantly improve stabilization performance and, in some cases, are indispensable for ensuring the desired stability behavior within prescribed time bounds. Consequently, the proposed framework provides both a novel theoretical perspective and a practical tool for the analysis and design of impulsive dynamical systems.

References

[1] F. Amato, G. de Tommasi, and A. Pironti, *Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems, Automatica*, 49(8), 2013, 2546–2550.

[2] W. M. Haddad and J. Lee, *Finite-Time Stabilization and Optimal Feedback Control for Nonlinear Discrete-Time Systems*, IEEE Transactions on Automatic Control, 2022.

[3] X. Li, D. W.C. Ho, and J. Cao, *Finite-time stability and settling-time estimation of nonlinear impulsive systems*, Automatica, 99, 2019, 361–368.

[4] H. Li., Ch. Li, T. Huang, and D. Ouyang, *Fixed-time stability and stabilization of impulsive dynamical systems*, Journal of the Franklin Institute, 354(18), 2017, 8626–8644.

[5] A. Nowakowski, *The dual dynamic programming*, Proceedings of the American Mathematical Society, 116(4), 1992, 1089–1096.

[6] A. Polyakov, *Nonlinear feedback design for fixed-time stabilization of linear control systems*, IEEE Transactions on Automatic Control, 57(8), 2012, 2106–2110.

[7] J. Wu, X. Li, X. Xie, *Finite-time stability for time-varying nonlinear impulsive systems*, Mathematical Methods in the Applied Sciences, 2021, 1–16.

[8] T. Yang, *Impulsive control theory*, Springer, 2001.

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