Multiple Solutions for the p-Laplacian Equation with Convection

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We investigate in [1] the following class of nonlinear elliptic problems involving a p-Laplacian operator and a gradient-dependent reaction term, which introduces a convective component and disrupts the variational structure of the problem, i.e.,

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

being $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain with smooth boundary and $1 < p < N$. On the reaction term, we assume that

(H) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, such that

(i) $f(x, 0, \xi) = 0$ for a.a $x \in \Omega$, for all $\xi \in \mathbb{R}^N$;

(ii) there exists $\ell \in [1, p]$, such that

$$|f(x, t, \xi)| \leq c(1 + |t|^{\ell-1})$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, being c a positive constant. In addition, if $\ell = p$ we require that $c < \lambda_1$, being λ_1 the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$;

(iii) there exist c_0, c_1 positive constants and $s \leq p \leq r < p^*$, $s \neq r$, such that

$$f(x, t, \xi)t \geq c_0|t|^s - c_1|t|^r$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$;

(iv) there exist $\zeta_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and constants c_-, c_+ such that

$$\zeta_-(x) \leq c_- < 0 < c_+ \leq \zeta_+(x) \text{ for a.a. } x \in \Omega;$$
$$f(x, \zeta_+(x), \xi) \leq f(x, \zeta_-(x), \xi) \text{ for a.a. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^N.$$

(v) there exist $\mu \in (1, \ell)$ and $\delta_0 > 0$ such that

$$0 < f(z, s, \xi)s \leq \mu F(x, s, \xi)$$

for a.a. $x \in \Omega$, for all $0 < |s| \leq \delta_0$, $x \in \mathbb{R}^N$ and $\operatorname{essinf}_{\Omega \times \mathbb{R}^N} F(\cdot, \delta_0, \cdot) > 0$.

Motivated by recent advances in the field, see e.g. [2, 3], we establish the existence of at least two weak solutions with sign information—one positive and one negative—without imposing asymptotic conditions near zero. Our approach relies on variational techniques on a counterpart of problem (P) with frozen gradient term, and the Leray–Schauder alternative principle, extending previous results and pertaining to a broader class of nonlinearities.

References

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